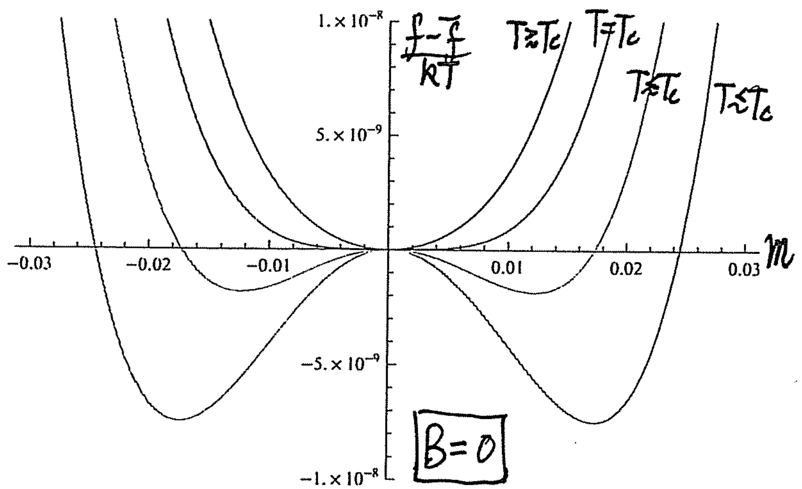
I. Looking at f as a function of m: A tiny twist leads to something big

From Eq. (14):

$$f-\bar{f} = \frac{Jzm^2}{2} - kT\ln\left[2\cosh\left(\frac{Jzm}{kT}+\frac{B}{kT}\right)\right] \quad (14) \text{ (Ising Model)}$$

add in a constant corresponding to $f(T\to\infty)$ $[-kT\ln 2]$ for completeness [doesn't matter]

It is useful to look at the behavior of $(f-\bar{f})$ as a function of m.

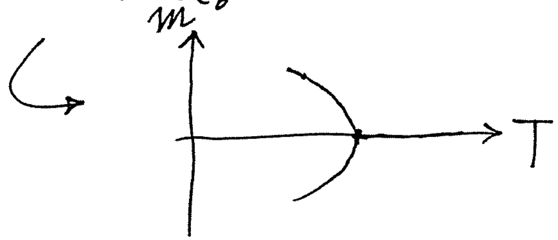


$T>T_c$: $m=0$ is a minimum

$T=T_c$: $m=0$ is a minimum and curve becomes rather flat near $m=0$

$T\lesssim T_c$: $m=0$ is a maximum and two other values ($\pm$) are minima

Tracing the minima:



Key Observations:

$f(m)$ changes qualitatively as T gets across T_c

$T > T_c$, $m = 0$ gives minimum f (order parameter $=0$, disordered phase)

$T \lesssim T_c$, $m \neq 0$ gives minimum f (order parameter $\neq 0$, disordered phase)

Ising Model

There is an explicit expression (Eq. (14))

$$f = \bar{f} + \frac{T_c}{2} k\, m^2 - kT \ln\left[2\cosh\left(\frac{T_c}{T}\cdot m\right)\right] \quad (B=0 \text{ case})$$

For $|m| \ll 1$ (i.e. near T_c), expand in powers of m (Ex.)

$$f = \underset{\substack{\sim m^0 \\ \text{term}}}{\bar{f}} + \underbrace{\frac{1}{2} k \frac{T_c}{T}(T - T_c)}_{\boxed{\text{prefactor } \sim (T-T_c) \text{ and changes sign as } T \text{ goes across } T_c}} \underset{\substack{m^2 \\ \text{term}}}{m^2} + \underbrace{\frac{1}{12} kT \left(\frac{T_c}{T}\right)^4}_{\text{pre-factor is positive}} \underset{\substack{m^4 \\ \text{term}}}{m^4} + (\text{terms of higher powers of } m)$$

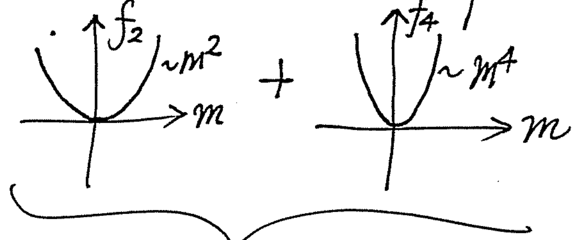
The key feature of mean field theory that gives the critical phenomena is:

- A free energy takes on the form as a function of the order parameter near the critical point: ($T \approx T_c$)

$$\boxed{f(m) = f_0 + \underbrace{a_2(T)}_{\sim (T-T_c)} m^2 + \overset{a_4>0}{a_4} m^4}^{\dagger} \quad (15)$$

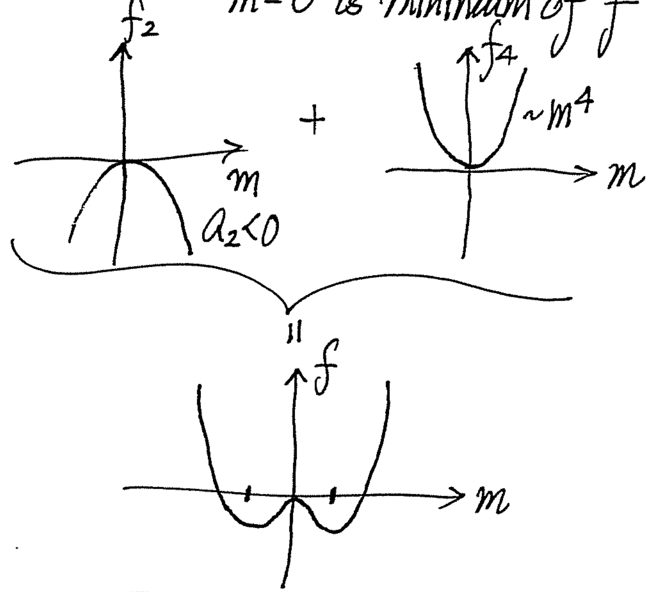
$\sim (T-T_c) \to$ +ve $T>T_c$; $\to$ −ve $T<T_c$

$T>T_c$: m^2 and m^4 terms are positive



$m=0$ is minimum of f

$T<T_c$:



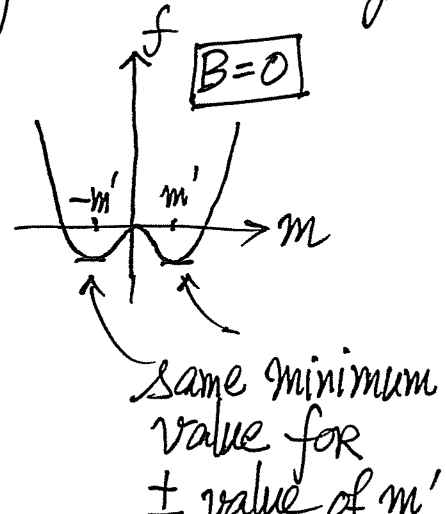
- Applicable to all problems
  [need to identify order parameter of a problem]
- Emphasized on math form of Eq. (15)
  [not microscopic detail but symmetry of problem]

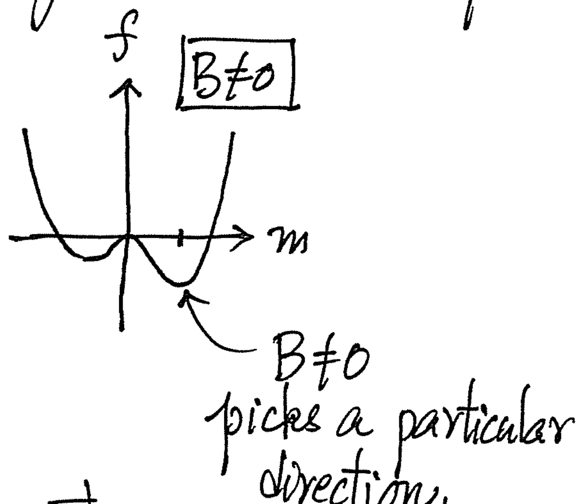
$\dagger$ This emprical form is the beginning of the Landau Theory of continuous phase transitions (Landau: 1937). Landau wrote down Eq. (15) by physical insight.

If $B \neq 0$, i.e. there is an applied field, go back to Eq.(14) and expand $\ln\left[2\cosh\left(\frac{Jzm}{kT} + \frac{B}{kT}\right)\right]$ for small argument, cross term of mB appears

$$\boxed{f(m) = f_0 \underbrace{- Bm}_{\text{due to applied field}} + a_2(T)\, m^2 + a_4 m^4} \quad (16)$$

Effect: depending on direction of B, it gives a bias to a particular direction





There is a discontinuous jump of m from $m=0$ to $m \neq 0$ [1st order transition]

J. Landau Theory of Continuous Phase Transitions (Optional)

Landau (1937)

- Introduced the idea of order parameter [free from any specific problems]
  - ferromagnets : magnetisation
  - Superconductors : fraction of electrons becoming Cooper pairs
  - Liquid crystals : Angle between director of molecule to alignment direction
  - Quasi-Crystals : Set of 5-fold symmetrical vectors

  Often given a problem, one needs to look for the proper order parameter
- Write free energy in powers of m, as $|m| \ll 1$ near critical point
- Powers of m reflect symmetry of system [Hamiltonian]

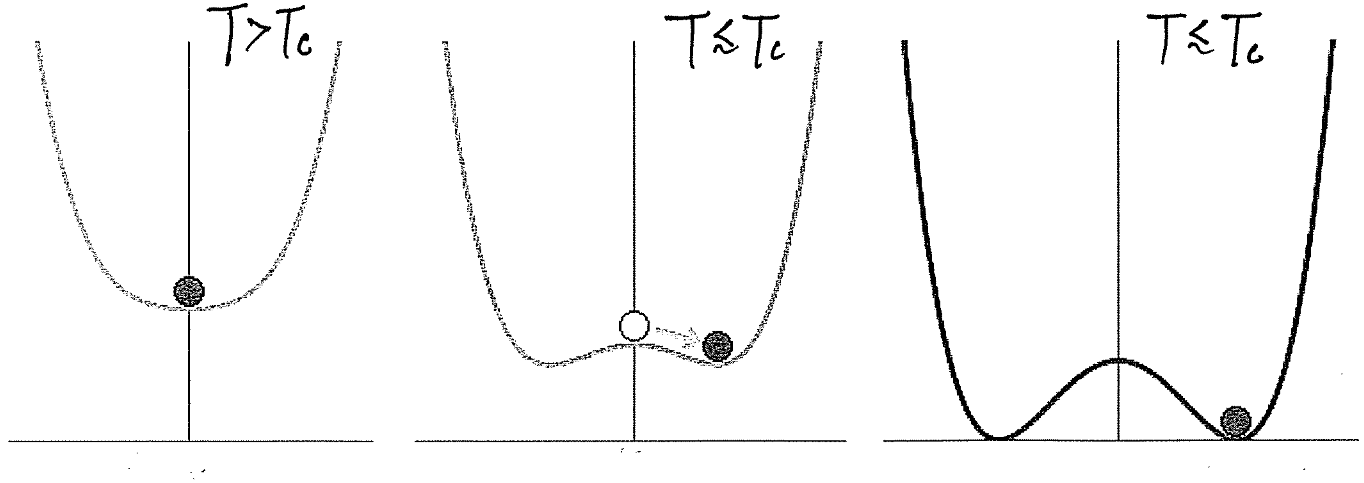
Then, Landau introduced (for $B=0$)

$$\boxed{f(m) = \underbrace{f_0}_{\sim m^0 \text{ term}} + \underbrace{a_2(T)}_{\substack{\sim (T-T_c)\,\& \\ \text{changes signs}}} m^2 + \underbrace{a_4}_{a_4>0} m^4} \tag{15}$$

as a <u>phenomenological</u> (唯像) theory of critical phenomena

Ising case: Only up & down. The Hamiltonian has up/down symmetry. The m^2 and m^4 terms do not distinguish directions, thus they go with the symmetry of Hamiltonian. The $T>T_c$ paramagnetic phase also respects the symmetry. But the $T<T_c$ ferromagnetic phase <u>has to pick a direction</u> as it forms!

Big idea: Attached to critical phenomena is the idea of <u>spontaneous symmetry breaking</u>!



System needs to pick a direction (side)
In doing so, there is the notion of
what is the "up" side and what is the "down" side.
A particular direction is chosen spontaneously, and
the "up/down" symmetry is gone (broken).

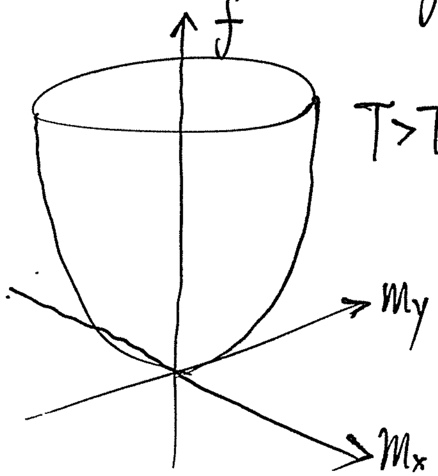
The idea is general! [Not only for Ising Model]

E.g. Each spin can point to <u>any direction</u> on a plane

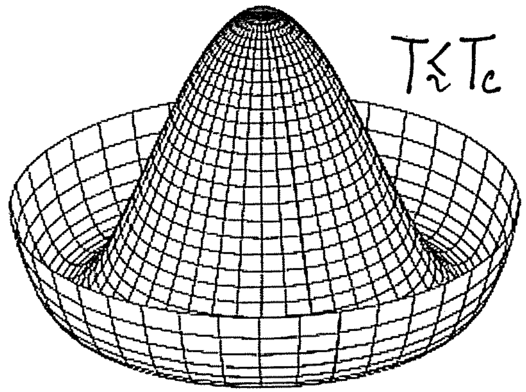
(XY model)

$T > T_c$: Paramagnetic phase has "circular" symmetry

$T < T_c$: System needs to select a direction



$T > T_c$



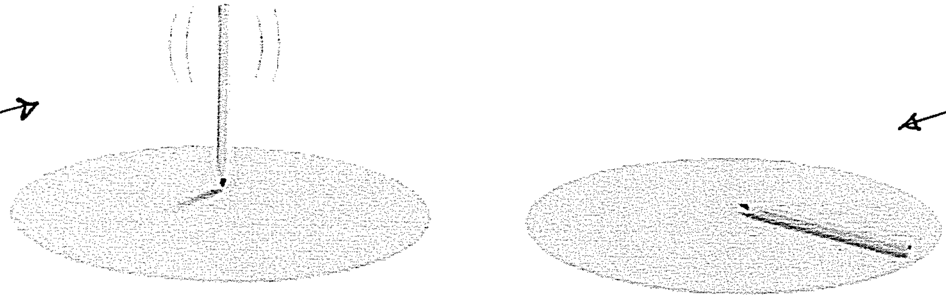
$T \lesssim T_c$

"shape of a Mexican hat"

There is a ring of possible states with $|m| \neq 0$.
But system needs to take a direction.

A usual analogy of this phenomenon is that of a fallen pencil.

All directions look the same



fallen pencil picks a direction

Spontaneous broken symmetry. The world of this pencil is completely symmetrical. All directions are exactly equal. But this symmetry is lost when the pencil falls over. Now only one direction holds. The symmetry that existed before is hidden behind the fallen pencil.

Marble on top of Mexican hat

Marble rolls down in a direction

All these phenomena come from a mathematical form:

$$f(m) = f_0 + a_2(T)\, m^2 + a_4\, m^4$$

or often written as:

$$f(\phi) = f_0 + a_2(T)\, \phi^2 + a_4\, \phi^4$$ where ϕ is the order parameter

As symmetry dictated the development of 20th century physics, the idea of spontaneous symmetry breaking played an important role in particle physics.+

$$
\begin{aligned}
L = & -\frac{1}{4}W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} \\
& +\overline{\psi}_j\gamma^\mu\left(i\partial_\mu - g\tau_j\cdot W_\mu - g'Y_jB_\mu - g_s\mathbf{T}_j\cdot\mathbf{G}_\mu\right)\psi_j \\
& +|D_\mu\phi|^2 + \mu^2|\phi|^2 - \lambda|\phi|^4 \\
& -\left(y_j\overline{\psi}_{jL}\phi\psi_{jR} + y'_j\overline{\psi}_{jL}\phi_c\psi_{jR} + \text{ conjugate}\right)
\end{aligned}
$$

Lagrangian Density of the Standard Model

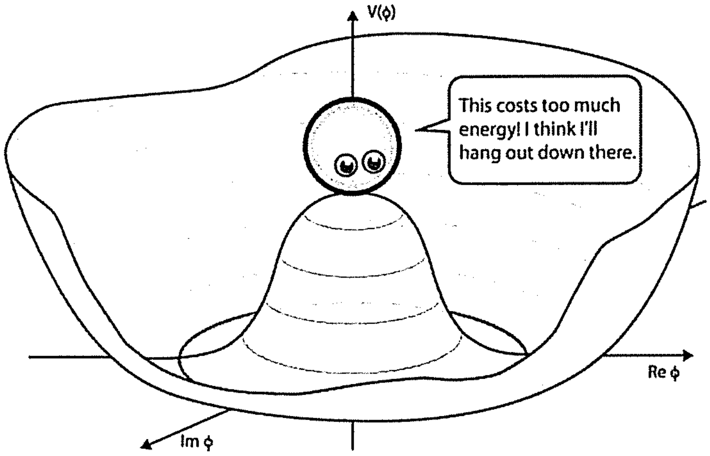
$$\left(|D_\mu\phi|^2 - \left[\underbrace{-\mu^2|\phi|^2 + \lambda|\phi|^4}_{\text{Mexican hat shape}}\right]\right)$$

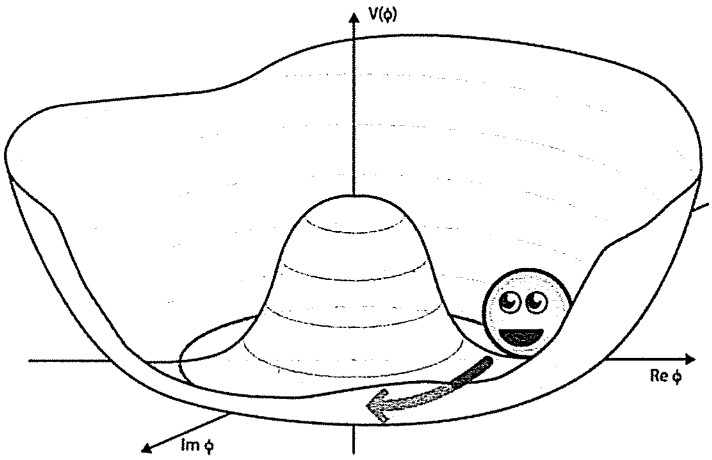
ϕ is the Higgs field

kinetic energy term (like $(\nabla\phi)^2$)

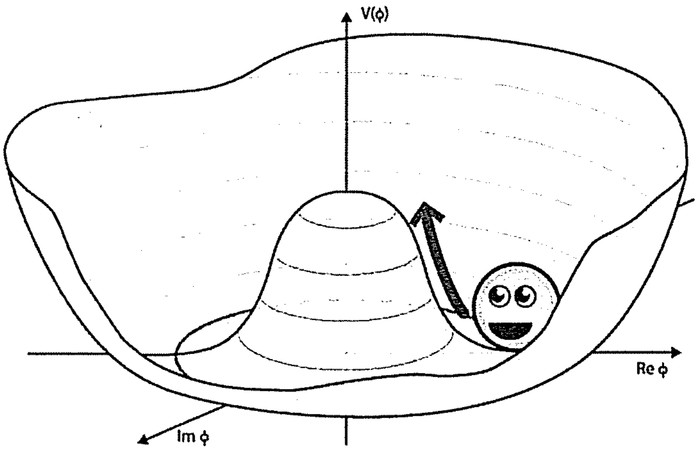
Mexican hat shape } so ϕ picks a direction (spontaneous symmetry breaking)

+ Nobel Prize (2008) to Nambu, Kobayashi, Maskawa for their work on spontaneous symmetry breaking in subatomic physics. The Higgs mechanism (Nobel Prize 2013) is also related to SSB.





Symmetry breaking
Higgs field picks a direction [real value]



Low-energy excitation
- massive
- massless

Source: http://cph-theory.persiangig.com/90.11.26-3.htm

See also Nobel Prize Announcements in 2008 and 2013.

$$
\begin{aligned}
L \;=\; & -\frac{1}{4}W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} \\
& + \overline{\psi}_j\gamma^\mu\left(i\partial_\mu - g\tau_j\cdot W_\mu - g'Y_jB_\mu - g_s\mathbf{T}_j\cdot\mathbf{G}_\mu\right)\psi_j \\
& + |D_\mu\phi|^2 + \mu^2|\phi|^2 - \lambda|\phi|^4 \\
& - \underbrace{\left(y_j\overline{\psi}_{jL}\phi\psi_{jR} + y_j'\overline{\psi}_{jL}\phi_c\psi_{jR} + \text{ conjugate}\right)}_{\text{coupling of Higgs field } \phi \text{ with leptons and quarks can lead to masses}}
\end{aligned}
$$

free Lagrangian for force carriers

matter interacts by exchanging force carriers

Higgs Mechanism

this term carries the force carriers ("gauge bosons") and give them masses

But all these started with the potential function

$$\underbrace{-\mu^2|\phi|^2 + \lambda|\phi|^4}_{\text{the Mexican hat!}}$$

<u>Summary</u>: Phase transitions and Critical Phenomena form a subject with rich physics. Interactions are essential. Yet they exhibit many universal behavior. The universality implies system details are not important for phenomena near the critical point.

<u>Refs</u>: For students who want to learn more on phase transitions and critical phenomena, see

- M. Gitterman, "Phase Transitions: Modern Applications" (World Scientific)
- J.M. Yeomans, "Statistical Mechanics of Phase Transitions" (Clarendon Press)
- K. Christensen & N.R. Moloney, "Complexity and Criticality" (Imperial College Press)